The concrete result in the three cases of adjustment appears to be this, that for a finite number of applications, the limiting form is approached more rapidly in the first case than in the second, and more rapidly in the second than in the third. Thus under the first case, p. 133, the coefficients for only 16 applications approach the limiting form more nearly than the coefficients for 32 applications do under the second case at p. 138, the limit being apparently farther off in the latter case.

NOTE.—When a series is adjusted by two formulas successively, as at p. 66 of ANALYST for May 1878, where the two are denoted by

$$u'_0 = l_0 u_0 + l_1(u_1+u_{-1}) + l_2(u_2+u_{-2}) + \dots + l_m(u_m+u_{-m}),$$
$$u''_0 = L_0 u'_0 + L_1(u'_1+u'_{-1}) + L_2(u'_2+u'_{-2}) + \dots + L_n(u'_n+u'_{-n}),$$

the coefficients in the resultant formula are the same as those which belong to the several powers of x in the product of the two polynomials

$$l_m + l_{m-1}x + l_{m-2}x^2 + \dots + l_0 x^m + l_1 x^{m+1} + \dots + l_m x^{2m},$$
$$L_n + L_{n-1}x + L_{n-2}x^2 + \dots + L_0 x^n + L_1 x^{n+1} + \dots + L_n x^{2n}.$$

This property, which I did not perceive at first, systematizes the matter and simplifies it to the mind, although it makes no difference in the actual amount of computation to be done, in forming the resultant coefficients.

Since the resultant for k successive adjustments by one formula has the same coefficients as the corresponding polynomial raised to the k power, we are enabled to demonstrate the proposition (3) at p. 129 of the ANALYST for Sept., 1878, by means of the Multinomial Theorem, as I will take occasion to show hereafter. The demonstration includes the general case where the adjustment formula is not symmetrical on each side of the middle, so that λ_r and λ_{-r} are not necessarily equal.

GENERAL RULES FOR THE FORMATION OF MAGIC SQUARES OF ALL ORDERS.

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THE following general rules for the formation of magic squares, whether of odd or even degree, may be new to the younger generation of mathematicians, and possibly to some others. The squares constructed by these

methods by no means exhaust all the possible arrangements, but they furnish squares in great number and variety when the degree is even. For the sake of completeness, we will begin by stating the rule (which is probably familiar to all) for producing squares when the number of cells on a side is odd.

CASE 1.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

We begin by placing 1 in the middle cell of the upper row and, using the natural series of numbers,* proceed always diagonally upward to the right, except when we come to the edge of the square, or to a cell already filled. When the number would fall outside of the square, carry it to the extreme cell in that row or column in which the outside cell occurs. When a cell is already filled, or when the right hand upper corner cell is reached, place the the number in the cell just below.

CASE 2. When the number on a side is even and of the form $4n$.

Imagine the square to be subdivided into squarelets of 4 cells each, the four central cells comprising one, and conceive these squarelets as of *two* kinds alternating with each other. Place 1 in the left upper corner cell and proceed horizontally to the right, filling the squarelets of *one* kind successively. When the end of one row is reached go to the extreme left cell of the next row and again advance to the right as before. For the 4-square the result of this operation is as follows:

1			4
	6	7	
	10	11	
13			16

Next begin with the right lower corner cell, considering 1 as falling on this, and proceed regularly to the left, filling the empty cells with the numbers belonging to them. The result of this second operation is as follows:

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

*It is hardly necessay to state that in forming squares of this kind, *any* arithmetical series may be used.

CASE 3. When the number of cells on a side is even and of the form $4n+2$.

Taking the 10-square for a particular example, first arrange the numbers in it as follows:

I.

1	1	2_1	3_2	4	5_3	6_3	7	8_2	9_1	10
2	11_3	12	13_1	14_2	15	16	17_2	18_1	19	20_3
3	21	22_3	23	24_1	25_2	26_2	27_1	28	29_3	30
4	31_2	32	33_3	34	35_1	36_1	37	38_3	39	40_2
5	41_1	42_2	43	44_3	45	46	47_3	48	49_2	50_1
6	51_1	52_2	53	54_3	55	56	57_3	58	59_2	60_1
7	61_2	62	63_3	64	65_1	66_1	67	68_3	69	70_2
8	71	72_3	73	74_1	75_2	76_2	77_1	78	79_3	80
9	81_3	82	83_1	84_2	85	86	87_2	88_1	89	90_3
10	91	92_1	93_2	94	95_3	96_3	97	98_2	99_1	100

Beginning at the left upper corner, designate the cells in the first row and first column respectively by the numbers from 1 to 10. Then we can denote any cell by the numbers standing at the beginning of the column and row in which it occurs. Thus, for instance, the cell containing 44 is denoted by (4, 5); that is, it occupies the 4th column and 5th row.

Next form the following auxiliary square with $2n+1$ rows and $2n+1$ columns:

II.

1 10	2 9	3 8	4 7	5 6
2 9	3 8	4 7	5 6	1 10
3 8	4 7	5 6	1 10	2 9
4 7	5 6	1 10	2 9	3 8
5 6	1 10	2 9	3 8	4 7

In the first column place the numbers from 1 to 5, and to their right in the same column the complements of these numbers to $4n+3$. In the top cell of the next column place 2, and descending write the numbers 1 to 5 with their respective complements, preserving the cyclic order of the first column. At the top of the next column place 3, and so on.

The numbers in the first column of II are identified with those of the first row of I; those of the remaining columns of II are identified with those of the first column of I. The numbers of I which are designated by the combination of one of the numbers in the 1st column of II with one of the two numbers of the same row in the *last* column, are indicated by

the suffix 1; those designated by a number of the 1st column of II combined with one in the column *next* the last, are indicated by the suffix 2; those designated by a number of the 1st column of II with a number of any *one* of the other columns except the last two are indicated by the suffix 3.

For example, in I, 51 has suffix 1 because it corresponds to the combination (1, 6) from the first and last columns of II; 76 has the suffix 2 because the combination (5, 8) is taken from the first and next to the last columns of II; and 68 has the suffix 3 because the combination (8, 7) is from the first and second columns of II. Four kinds of numbers are thus distinguished: (1), those without any suffix, which keep their original places in the magic square III; (2), those with suffix 1, any corresponding four of which—that is any two pair equidistant from the sides—are interchanged in the following manner: the arrangement

$\begin{array}{ccc} a & \cdot\cdot & b \\ : & & : \\ c & \cdot\cdot & d \end{array}$ becomes $\begin{array}{ccc} d & \cdot\cdot & c \\ : & & : \\ a & \cdot\cdot & b \end{array}$; (3), those with suffix 2 are interch'd thus: $\begin{array}{ccc} a & \cdot\cdot & b \\ : & & : \\ c & \cdot\cdot & d \end{array}$ becomes $\begin{array}{ccc} d & \cdot\cdot & a \\ : & & : \\ b & \cdot\cdot & c. \end{array}$

(Instead of the change in (2) we may substitute $\begin{array}{ccc} c & \cdot\cdot & d \\ : & & : \\ a & \cdot\cdot & b \end{array}$, and for the one in (3), $\begin{array}{ccc} b & \cdot\cdot & c \\ : & & : \\ a & \cdot\cdot & d \end{array}$). For exam.,

$\begin{array}{ccc} 35 & \cdot\cdot & 36 \\ : & & : \\ 65 & \cdot\cdot & 66 \end{array}$ becomes $\begin{array}{ccc} 66 & \cdot\cdot & 65 \\ : & & : \\ 35 & \cdot\cdot & 36 \end{array}$; and $\begin{array}{ccc} 3 & \cdot\cdot & 8 \\ : & & : \\ 93 & \cdot\cdot & 98 \end{array}$ becomes $\begin{array}{ccc} 98 & \cdot\cdot & 3 \\ : & & : \\ 8 & \cdot\cdot & 93 \end{array}$. (4). The numbers with suffix 3 are arranged as the second class in the previous case ($4n$).

III.

1	99	98	4	96	95	7	3	92	10
90	12	88	87	15	16	14	83	19	81
21	79	23	77	76	25	74	28	72	30
70	32	68	34	66	65	37	63	39	41
60	59	43	59	45	46	54	48	42	61
41	49	53	47	55	56	44	58	52	50
40	62	38	64	35	36	67	33	69	61
71	29	73	24	26	75	27	78	22	80
20	82	13	17	85	86	84	18	89	11
91	2	2	94	6	5	97	93	9	100

The foregoing rules are condensed from an article in Hoffman and Natani's *Mathematisches Wörterbuch*, Vol. V.

The following is another set of rules for even squares, abridged from a pamphlet by S. M. Drach, F. R. A. S.

1°. When the side cells $= 4n$. Put $c = 16n^2 - 8i$, where i is 0, 1, 2, 3, . . . n^2—1. These substituted successively in this squarelet

$$\begin{array}{llll} 8i+1, & c-1, & c-2, & 8i+4, \\ c-4, & 8i+6, & 8i+7, & c-7, \\ 8i+8, & c-6, & c-5, & 8i+5, \\ c-3, & 8i+3, & 8i+2, & c, \end{array}$$

give n^2 squarelets, all *equal* magic squares; so that they may be arranged any how n in a row, or turned each on its centre, and still give the same linear sum $= 32n^2+2n$.

2°. When the side cells $= 4n+2$. Fill up the $4n$-square by the previous rule and then add $8n+2$ to each of its numbers. Surround this last square by a mono-celular border of which the upper and lower rows are respectively

$$\begin{array}{l|c|c|c|c|c|c|c|} 1 & s-2 & 8n+2 & cp & & cp & 6n+4 & \\ & & & & \cdots\cdots & & & \text{omit } 6n+3 \\ s-4n-2 & cp & cp & 8n+1 & & 6n+5 & cp & \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|l} cp & cp & 6n & & 4n+4 & cp & 4n+2 & \\ & & & \cdots\cdots & & & & \\ 6n+2 & 6n+1 & cp & & cp & 4n+3 & s-1 & , \end{array}$$

where $s = [(4n+2)^2+1]$, and cp = the complement to s of the opposite number. The vertical rows are,

left, $\quad 1,\ s-3,\ 4,\ s-5, \ldots s-4n-1,\ 6n+3,\ s-4n-2,$

right $\quad 4n+2,\ 3,\ s-4,\ 5, \ldots\ 4n+1,\ s-6n-3,\ s-1.$

Hence to form the border, place 1 in N. W. corner, $s-1$ at S. E. corner. $4n+2$ at N. E. corner, and $s-4n-2$ at S. W. corner, next to which put 2; and place $4n+3$, $4n+4$, up to $6n+1$, alternately from bottom, leftward; place $6n+2$ next $6n+1$ (omit $6n+3$), $6n+4$ at top, $6n+5$ at bottom, and so on; complete s in opposite cells.

For verticals, under $4n+2$ put 3, 4 opposite, 5 E., 6 W., 7 E., &c., to $(4n+1)$ E., $(6n+3)$ W.; their complements to s in the opposite cells, and it is done.

Any one wishing to go deeper into this subject will find it treated in a highly general and rigorous manner in a work entitled *Die Magishen Quadrate*, by Theodore Hugel.

According to this author the 3-square admits of 1 arrangement, the 4-square of 24, the 5-square of 6624, the 8-square of 834075520, the 11-sq, of 11736148838400000, &c.

According to an approximate calculation, the paper that would be required to contain all the squares with 13 cells on a side would cover the whole surface of the earth about 348 times.